

# Permutation Complexity Related to the Letter Doubling Map

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Given a countable set  $X$  (usually taken to be  $\mathbb{N}$  or  $\mathbb{Z}$ ), an infinite permutation  $\pi$  of  $X$  is a linear ordering  $\prec_\pi$  of  $X$ , introduced in [6]. This paper investigates the combinatorial complexity of infinite permutations on  $\mathbb{N}$  associated with the image of uniformly recurrent aperiodic binary words under the letter doubling map. An upper bound for the complexity is found for general words, and a formula for the complexity is established for the Sturmian words and the Thue-Morse word.

## 1 Introduction

Permutation complexity of aperiodic words is a relatively new notion of word complexity which is based on the idea of an infinite permutation associated to an aperiodic word. For an infinite aperiodic word  $\omega$ , no two shifts of  $\omega$  are identical. Thus, given a linear order on the symbols used to compose  $\omega$ , no two shifts of  $\omega$  are equal lexicographically. The infinite permutation associated with  $\omega$  is the linear order on  $\mathbb{N}$  induced by the lexicographic order of the shifts of  $\omega$ . The permutation complexity of the word  $\omega$  will be the number of distinct subpermutations of a given length of the infinite permutation associated with  $\omega$ .

We start with some basic notation and definitions. Some properties of infinite permutations are given in Section 2. In Section 3 we introduce a mapping,  $\delta$ , on the set of subpermutations of an uniformly recurrent word, and an upper bound for the complexity function is calculated for the image of an aperiodic uniformly recurrent word under the letter doubling map. We then show that when the mapping  $\delta$  is injective it implies that restricting an image of  $\delta$  is also injective in Section 4. The complexity function is established for the image of a Sturmian word in Section 5, and for the image of the Thue-Morse word in Section 6.

### 1.1 Permutations from words

In this writing a *word* over  $\mathcal{A}$  will be a right infinite sequence of symbols of the form  $\omega = \omega_0\omega_1\omega_2\ldots$  with each  $\omega_i \in \mathcal{A}$ , and the set of all words over  $\mathcal{A}$  is denoted  $\mathcal{A}^\mathbb{N}$ . A *finite word* over  $\mathcal{A}$  is a word of the form  $u = a_1a_2\ldots a_n$  with  $n \geq 0$  (if  $n = 0$  we say  $u$  is the *empty word*, denoted  $\varepsilon$ ) and each  $a_i \in \mathcal{A}$ , with the set of all finite words over the alphabet  $\mathcal{A}$  is denoted by  $\mathcal{A}^*$ . The *length* of the word  $u$  is the number of symbols in the sequence and is denoted by  $|u| = n$ . For  $a \in \mathcal{A}$ , let  $|u|_a$  denote the number of occurrences of the letter  $a$  in the word  $u$ .

Any word of the form  $u = \omega_i\omega_{i+1}\ldots\omega_{i+n-1}$ , with  $i \geq 0$ , is called a *factor* of  $\omega$  of length  $n \geq 1$ . The set of all factors of a word  $\omega$  is denoted by  $\mathcal{F}(\omega)$ . The set of all factors of length  $n$  of  $\omega$  is denoted  $\mathcal{F}_\omega(n)$ , and let  $\rho_\omega(n) = |\mathcal{F}_\omega(n)|$ . The function  $\rho_\omega : \mathbb{N} \rightarrow \mathbb{N}$  is called the *factor complexity function* of  $\omega$  and it counts the number of factors of length  $n$  of  $\omega$ . For a natural number  $i$  we denote by

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$\omega[i] = \omega_i \omega_{i+1} \omega_{i+2} \omega_{i+3} \dots$  the  $i$ -letter shift of  $\omega$ . For natural numbers  $i \leq j$ ,  $\omega[i, j] = \omega_i \omega_{i+1} \omega_{i+2} \dots \omega_j$  denotes the factor of length  $j - i + 1$  starting at position  $i$  in  $\omega$ .

For words  $u \in A^*$  and  $v \in A^* \cup A^\mathbb{N}$  where  $\omega = uv$ , we call  $u$  a *prefix* of  $\omega$  and  $v$  a *suffix* of  $\omega$ . A word  $\omega$  is said to be *periodic* of period  $p$  if for each  $i \in \mathbb{N}$ ,  $\omega_i = \omega_{i+p}$ , and  $\omega$  is said to be *eventually periodic* of period  $p$  if there exists an  $N \in \mathbb{N}$  so that for each  $i > N$ ,  $\omega_i = \omega_{i+p}$ ; or equivalently,  $\omega$  has a periodic suffix. A word  $\omega$  is said to be *aperiodic* if it is not periodic or eventually periodic.

The infinite word  $\omega \in \mathcal{A}^\mathbb{N}$  is said to be *recurrent* if for any prefix  $p$  of  $\omega$  there exists a prefix  $q$  of  $\omega$  so that  $q = pvp$  for some  $v \in \mathcal{A}^*$ . Equivalently, a word  $\omega$  is recurrent if each factor of  $\omega$  occurs infinitely often in  $\omega$ . The word  $\omega \in \mathcal{A}^\mathbb{N}$  is *uniformly recurrent* if each factor occurs infinitely often with bounded gaps. Thus if  $\omega$  is uniformly recurrent, for each integer  $n > 0$  there is a positive integer  $N$  so that for each factor  $v$  of  $\omega$  with  $|v| = N$ ,  $\mathcal{F}_\omega(n) \subset \mathcal{F}(v)$ .

A morphism on  $\mathcal{A}$  is a map  $\varphi : \mathcal{A}^* \rightarrow \mathcal{A}^*$  so that  $\varphi(uv) = \varphi(u)\varphi(v)$  for any  $u, v \in \mathcal{A}^*$ . The morphism  $d : \mathcal{A}^* \mapsto \mathcal{A}^*$  defined by  $d(a) = aa$  for each  $a \in \mathcal{A}$  is called the *letter doubling map*.

The idea of an infinite permutation that will be here used was introduced in [6]. This paper will be dealing with permutation complexity of infinite words so the set used in the following definition will be  $\mathbb{N}$  rather than an arbitrary countable set. To define an *infinite permutation*  $\pi$ , start with a total order  $\prec_\pi$  on  $\mathbb{N}$ , together with the usual order  $<$  on  $\mathbb{N}$ . To be more specific, an infinite permutation is the ordered triple  $\pi = \langle \mathbb{N}, \prec_\pi, < \rangle$ , where  $\prec_\pi$  and  $<$  are total orders on  $\mathbb{N}$ . The notation to be used here will be  $\pi(i) < \pi(j)$  rather than  $i \prec_\pi j$ .

Given an infinite aperiodic word  $\omega = \omega_0 \omega_1 \omega_2 \dots$  on an alphabet  $\mathcal{A}$ , fix a linear order on  $\mathcal{A}$ . We will use the binary alphabet  $\mathcal{A} = \{0, 1\}$  and use the natural ordering  $0 < 1$ . Once a linear order is set on the alphabet, we can then define an order on the natural numbers based on the lexicographic order of shifts of  $\omega$ . Considering two shifts of  $\omega$  with  $a \neq b$ ,  $\omega[a] = \omega_a \omega_{a+1} \omega_{a+2} \dots$  and  $\omega[b] = \omega_b \omega_{b+1} \omega_{b+2} \dots$ , we know that  $\omega[a] \neq \omega[b]$  since  $\omega$  is aperiodic. Thus there exists some minimal number  $c \geq 0$  so that  $\omega_{a+c} \neq \omega_{b+c}$  and for each  $0 \leq i < c$  we have  $\omega_{a+i} = \omega_{b+i}$ . We call  $\pi_\omega$  the infinite permutation associated with  $\omega$  and say that  $\pi_\omega(a) < \pi_\omega(b)$  if  $\omega_{a+c} < \omega_{b+c}$ , else we say that  $\pi_\omega(b) < \pi_\omega(a)$ .

For natural numbers  $a \leq b$  consider the factor  $\omega[a, b] = \omega_a \omega_{a+1} \dots \omega_b$  of  $\omega$  of length  $b - a + 1$ . Denote the finite permutation of  $\{1, 2, \dots, b - a + 1\}$  corresponding to the linear order by  $\pi_\omega[a, b]$ . That is  $\pi_\omega[a, b]$  is the permutation of  $\{1, 2, \dots, b - a + 1\}$  so that for each  $0 \leq i, j \leq (b - a)$ ,  $\pi_\omega[a, b](i) < \pi_\omega[a, b](j)$  if and only if  $\pi_\omega(a + i) < \pi_\omega(a + j)$ . Say that  $p = p_0 p_1 \dots p_n$  is a (*finite*) *subpermutation* of  $\pi_\omega$  if  $p = \pi_\omega[a, a + n]$  for some  $a, n \geq 0$ . For the subpermutation  $p = \pi_\omega[a, a + n]$  of  $\{1, 2, \dots, n + 1\}$ , we say the *length* of  $p$  is  $n + 1$ .

Denote the set of all subpermutations of  $\pi_\omega$  by  $\text{Perm}^\omega$ , and for each positive integer  $n$  let

$$\text{Perm}^\omega(n) = \{ \pi_\omega[i, i + n - 1] \mid i \geq 0 \}$$

denote the set of distinct finite subpermutations of  $\pi_\omega$  of length  $n$ . The *permutation complexity function* of  $\omega$  is defined as the total number of distinct subpermutations of  $\pi_\omega$  of a length  $n$ , denoted  $\tau_\omega(n) = |\text{Perm}^\omega(n)|$ .

**Example** Let's consider the well-known Fibonacci word,

$$t = 0100101001001010010100100101 \dots,$$

with the alphabet  $\mathcal{A} = \{0, 1\}$  ordered as  $0 < 1$ . We can see that  $t[2] = 001010 \dots$  is lexicographically less than  $t[1] = 100101 \dots$ , and thus  $\pi_t(2) < \pi_t(1)$ .

Then for a subpermutation, consider the factor  $t[3, 5] = 010$ . We see that  $\pi_t[3, 5] = (231)$  because in lexicographic order we have  $\pi_t(5) < \pi_t(3) < \pi_t(4)$ .

Infinite permutations associated with infinite aperiodic words over a binary alphabet act fairly well-behaved, but many of the arguments used for binary words break down when used with words over more than two letters. Given a subpermutation of length  $n$  of an infinite permutation associated with a binary word, a portion of length  $n - 1$  of the word can be recovered from the subpermutation. This is not always the case for subpermutations associated with words over 3 or more letters. For binary words the subpermutations depend on the order on the symbols used to compose  $\omega$ , but the permutation complexity does not depend on the order. For words over 3 or more letters, not only do the subpermutations depend on the order on the alphabet but so does the permutation complexity.

## 2 Some General Permutation Properties

Initially work has been done with infinite binary words (see [2, 6, 9, 10, 11]). Suppose  $\omega = \omega_0\omega_1\omega_2\ldots$  is an aperiodic infinite word over the alphabet  $\mathcal{A} = \{0, 1\}$ . First let's look at some remarks about permutations generated by binary words where we use the natural order on  $\mathcal{A}$ .

**Claim 2.1** ([9]) *For an infinite aperiodic word  $\omega$  over  $\mathcal{A} = \{0, 1\}$  with the natural ordering we have:*

- (1)  $\pi_\omega(i) < \pi_\omega(i+1)$  if and only if  $\omega_i = 0$ .
- (2)  $\pi_\omega(i) > \pi_\omega(i+1)$  if and only if  $\omega_i = 1$ .
- (3) If  $\omega_i = \omega_j$ , then  $\pi_\omega(i) < \pi_\omega(j)$  if and only if  $\pi_\omega(i+1) < \pi_\omega(j+1)$

**Lemma 2.2** ([9]) *Given two infinite binary words  $u = u_0u_1\ldots$  and  $v = v_0v_1\ldots$  with  $\pi_u[0, n+1] = \pi_v[0, n+1]$ , it follows that  $u[0, n] = v[0, n]$ .*

We do have a trivial upper bound for  $\tau_\omega(n)$  being the number of permutations of length  $n$ , which is  $n!$ . Lemma 2.2 directly implies a lower bound for the permutation complexity for a binary aperiodic word  $\omega$ , namely the factor complexity of  $\omega$ . Thus, initial bounds on the permutation complexity can be seen to be:

$$\rho_\omega(n-1) \leq \tau_\omega(n) \leq n!$$

For  $a \in \mathcal{A} = \{0, 1\}$ , let  $\bar{a}$  denote the *complement* of  $a$ , that is  $\bar{0} = 1$  and  $\bar{1} = 0$ . If  $u = u_1u_2u_3\ldots$  is a word over  $\mathcal{A}$ , the *complement* of  $u$  is defined to be the word composed of the complement of the letters in  $u$ , that is  $\bar{u} = \bar{u}_1\bar{u}_2\bar{u}_3\ldots$ . The following lemma shows the relationship of the complexity function between an aperiodic binary word  $\omega$  and its complement  $\bar{\omega}$ . This lemma will be used when calculating the permutation complexity of the image of Sturmian words under the doubling map in Section 5.

**Lemma 2.3** *Let  $\omega = \omega_0\omega_1\omega_2\ldots$  be an infinite aperiodic binary word, and let  $\bar{\omega} = \bar{\omega}_0\bar{\omega}_1\bar{\omega}_2\ldots$  be the complement of  $\omega$ . For each  $n \geq 1$ ,*

$$\tau_\omega(n) = \tau_{\bar{\omega}}(n).$$

We would like to define some terms that will be used repeatedly in this paper.

**Definition** Two permutations  $p$  and  $q$  of  $\{1, 2, \ldots, n\}$  have the *same form* if for each  $i = 0, 1, \ldots, n-1$ ,  $p_i < p_{i+1}$  if and only if  $q_i < q_{i+1}$ . For a binary word  $u$  of length  $n-1$ , say that  $p$  has *form*  $u$  if

$$p_i < p_{i+1} \iff u_i = 0$$

for each  $i = 0, 1, \ldots, n-2$ .

**Definition** Let  $p = \pi[a, a+n]$  be a subpermutation of the infinite permutation  $\pi$ . The *left restriction* of  $p$ , denoted by  $L(p)$ , is the subpermutation of  $p$  so that  $L(p) = \pi[a, a+n-1]$ . The *right restriction* of  $p$ , denoted by  $R(p)$ , is the subpermutation of  $p$  so that  $R(p) = \pi[a+1, a+n]$ . The *middle restriction* of  $p$ , denoted by  $M(p)$ , is the subpermutation of  $p$  so that  $M(p) = R(L(p)) = L(R(p)) = \pi[a+1, a+n-1]$ .

For each  $i$ , there are  $p_i - 1$  terms in  $p$  that are less than  $p_i$  and there are  $n - p_i$  terms that are greater than  $p_i$ . Thus consider some  $0 \leq i \leq n-1$  and the values of  $L(p)_i$  and  $R(p)_i$ . If  $p_0 < p_{i+1}$  there will be  $p_{i+1} - 2$  terms in  $R(p)$  less than  $R(p)_i$  so we have  $R(p)_i = p_{i+1} - 1$ . In a similar sense, if  $p_n < p_i$  we have  $L(p)_i = p_i - 1$ . If  $p_0 > p_{i+1}$  there will be  $p_{i+1} - 1$  terms in  $R(p)$  less than  $R(p)_i$  so we have  $R(p)_i = p_{i+1}$ . In a similar sense, if  $p_n > p_i$  we have  $L(p)_i = p_i$ .

The values in  $M(p)$  can be found by finding the values in  $R(L(p))$  or  $L(R(p))$ . Since  $R(L(p))$  or  $L(R(p))$  correspond to the same subpermutation of  $p$ ,  $R(L(p))_i < R(L(p))_j$  if and only if  $L(R(p))_i < L(R(p))_j$ . Therefore  $R(L(p)) = L(R(p))$ .

It should also be clear that if there are two subpermutations  $p = \pi_T[a, a+n]$  and  $q = \pi_T[b, b+n]$  so that  $p = q$  then  $L(p) = L(q)$ ,  $R(p) = R(q)$ , and  $M(p) = M(q)$ .

### 3 Uniformly Recurrent Words

Let  $\omega$  be an aperiodic infinite uniformly recurrent word over  $\mathcal{A} = \{0, 1\}$ , and  $\pi_\omega$  be the infinite permutation associated with  $\omega$  using the natural order on the alphabet. We would like to describe the infinite permutation associated with  $d(\omega)$ , the image of  $\omega$  under the doubling map. In this section we will define a mapping from the set of subpermutations of  $\pi_\omega$  onto the subpermutations of  $\pi_{d(\omega)}$ , and we will find an upper bound for the permutation complexity of the image of a uniformly recurrent aperiodic binary word.

Since  $\omega$  is a uniformly recurrent word it will not contain arbitrarily long strings of contiguous 0 or 1. Thus there are  $k_0, k_1 \in \mathbb{N}$  so that  $10^{k_0}1$  and  $01^{k_1}0$  are factors of  $\omega$ , but  $0^{k_0+1}$  and  $1^{k_1+1}$  are not. We then define the following classes of words:

$$\begin{aligned} C_0 &= 0^{k_0} \\ C_1 &= 0^{k_0-1}1 \\ &\vdots \\ C_{k_0-1} &= 01 \\ C_{k_0} &= 10 \\ C_{k_0+1} &= 1^20 \\ &\vdots \\ C_{k_0+k_1-1} &= 1^{k_1}. \end{aligned}$$

For each  $i \in \mathbb{N}$ ,  $\omega[i] = \omega_i \omega_{i+1} \dots$  can have exactly one of the above classes of words as a prefix. It should be clear  $C_0 < C_1 < \dots < C_{k_0+k_1-1}$ , and so  $d(C_i) < d(C_j)$  for  $i < j$  since the doubling map  $d$  is order preserving, as shown in Lemma 3.1. The next lemma will not only show that the doubling map is an order preserving map, but also the order of the image of  $\omega_i$  under the doubling map.

**Lemma 3.1** *Let  $\omega$  be as above. Suppose  $\omega[a]$  and  $\omega[b]$  are two shifts of  $\omega$  for some  $a \neq b$  so that  $\omega[a] < \omega[b]$ . Moreover, suppose  $C_i$  is a prefix of  $\omega[a]$  and  $C_j$  is a prefix of  $\omega[b]$  where  $i \leq j$ . Then  $d(\omega[a]) < d(\omega[b])$ , and*

- (a) If  $\omega_a = \omega_b = 0$  and  $i < j$ , then  $d(\omega)[2a] < d(\omega)[2a+1] < d(\omega)[2b] < d(\omega)[2b+1]$ .
- (b) If  $\omega_a = \omega_b = 0$  and  $i = j$ , then  $d(\omega)[2a] < d(\omega)[2b] < d(\omega)[2a+1] < d(\omega)[2b+1]$ .
- (c) If  $\omega_a = 0$  and  $\omega_b = 1$ , then  $d(\omega)[2a] < d(\omega)[2a+1] < d(\omega)[2b+1] < d(\omega)[2b]$ .
- (d) If  $\omega_a = \omega_b = 1$  and  $i < j$ , then  $d(\omega)[2a+1] < d(\omega)[2a] < d(\omega)[2b+1] < d(\omega)[2b]$ .
- (e) If  $\omega_a = \omega_b = 1$  and  $i = j$ , then  $d(\omega)[2a+1] < d(\omega)[2b+1] < d(\omega)[2a] < d(\omega)[2b]$ .

For  $k = \sup\{k_0, k_1\}$ , there is an  $N_k$  so any factor  $u$  of  $\omega$  of length  $n \geq N_k$  will contain all factors of length  $k$  as a subword, and so  $u$  will have  $C_j$  as a subword for each  $j$ . One note about the factors of  $d(\omega)$ . For  $n \geq N_k$  and two factors  $u = d(\omega)[2x, 2x+2n]$  and  $v = d(\omega)[2y+1, 2y+2n+1]$  of  $d(\omega)$ , then  $u \neq v$ . This is because a prefix of  $u$  will begin with an even number of one letter (either  $0^{2m}1$  or  $1^{2m}0$  for some  $m$ ), and a prefix of  $v$  will begin with an odd number of one letter (either  $0^{2m+1}1$  or  $1^{2m+1}0$  for some  $m$ ).

Fix a factor  $u$  of  $\omega$  of length  $n \geq N_k$ . There is an  $a$  so that  $u = \omega[a, a+n-1]$ . For each  $0 \leq i \leq n-1$  there is one  $j$  so that  $\omega[a+i]$  has  $C_j$  as a prefix. In the factor  $\omega[a, a+n+k-2]$  of length  $n+k-1$ , we will know explicitly which  $C_j$  is a prefix of the shift  $\omega[a+i]$  for each  $0 \leq i \leq n-1$ . Let  $p = \pi_\omega[a, a+n+k-1]$  be a subpermutation of  $\pi_\omega$  of length  $n+k$ . The factor  $\omega[a, a+n+k-2]$  of length  $n+k-1$  is the form of  $p$ , and has  $u$  as a prefix.

For each  $j \in \{0, 1, \dots, k_0 + k_1 - 1\}$  define

$$\gamma_j = \{ 0 \leq i \leq n-1 \mid C_j \text{ is a prefix of } \omega[a+i] \}.$$

So  $|\gamma_0| + |\gamma_1| + \dots + |\gamma_{k_0+k_1-1}| = n$  and  $\gamma_i \cap \gamma_j = \emptyset$  for  $i \neq j$ . Since  $|u| \leq N_k$ , we know  $|\gamma_j| \geq 1$  for each  $j$ . We can see  $d(u) = d(\omega)[2a, 2a+2n-1]$ , and let  $p'$  be the subpermutation  $p' = \pi_{d(\omega)}[2a, 2a+2n-1]$ . Using Lemma 3.1 and the size of each of the  $\gamma_j$  sets we can determine the values of  $p'$  based on the values of  $L^k(p)$ , the  $k$ -left restriction of  $p$ . For each  $j \in \{0, 1, \dots, k_0 + k_1 - 1\}$  define

$$S_j = \sum_{i=0}^j |\gamma_i|$$

and say  $S_{-1} = 0$ .

**Proposition 3.2** *Let  $\omega$ ,  $u$ ,  $p$ , and  $p'$  be as above. For each  $0 \leq i \leq n-1$ , there is a  $j$  so  $\omega[a+i]$  has  $C_j$  as a prefix.*

- (a) If  $p_i < p_{i+1}$  then  $p'_{2i} = L^k(p)_i + S_{j-1}$  and  $p'_{2i+1} = L^k(p)_i + S_j$
- (b) If  $p_i > p_{i+1}$  then  $p'_{2i} = L^k(p)_i + S_j$  and  $p'_{2i+1} = L^k(p)_i + S_{j-1}$

**Corollary 3.3** *Let  $\omega$  be as defined above. If  $\pi_\omega[a, a+n+k-1]$  and  $\pi_\omega[b, b+n+k-1]$ ,  $a \neq b$ , are subpermutations of  $\pi_\omega$  where  $\pi_\omega[a, a+n-1] = \pi_\omega[b, b+n-1]$  and for each  $0 \leq i \leq n-1$ , there is some  $j$  so that both  $\omega[a+i]$  and  $\omega[b+i]$  have  $C_j$  as a prefix. Then  $\pi_{d(\omega)}[2a, 2a+2n-1] = \pi_{d(\omega)}[2b, 2b+2n-1]$ .*

Fix a subpermutation  $p = \pi_\omega[a, a+n+k-1]$ , and let  $p' = \pi_{d(\omega)}[2a, 2a+2n-1]$ . The terms of  $p'$  can be defined using the method given in Proposition 3.2. Let  $q = \pi_\omega[b, b+n+k-1]$ ,  $b \neq a$ , be a subpermutation of  $\pi_\omega$  and let  $q' = \pi_{d(\omega)}[2b, 2b+2n-1]$  as in Proposition 3.2. The following lemma shows that if  $p = q$  we know  $p' = q'$ , but the converse of this is not necessarily true. The objective here is using the idea of  $p'$  to define a map from the set of subpermutations of  $\pi_\omega$  to the set of subpermutations of  $\pi_{d(\omega)}$ , and this map will be well-defined by Proposition 3.2.

**Lemma 3.4** *If  $p = q$ , then  $p' = q'$ .*

Thus there is a well-defined function from the set of subpermutations of  $\pi_\omega$  to the set of subpermutations of  $\pi_{d(\omega)}$ . Let  $p = \pi_\omega[a, a + n + k - 1]$ , and define  $\delta(p) = p' = \pi_{d(\omega)}[2a, 2a + 2n - 1]$  using the formula in Proposition 3.2. Thus we have the map

$$\delta : \text{Perm}^\omega(n + k) \rightarrow \text{Perm}^{d(\omega)}(2n)$$

Not all subpermutations of  $\pi_\omega$  will be the image under  $\delta$  of another subpermutation.

Let  $n > 2N_k$  and  $a$  be natural numbers. Then  $n$  and  $a$  can be either even or odd, and for the subpermutation  $\pi_{d(\omega)}[a, a + n - 1]$ , there exist natural numbers  $b$  and  $m$  so that one of 4 cases hold:

1.  $\pi_{d(\omega)}[a, a + n] = \pi_{d(\omega)}[2b, 2b + 2m]$ , even starting position with odd length.
2.  $\pi_{d(\omega)}[a, a + n] = \pi_{d(\omega)}[2b, 2b + 2m - 1]$ , even starting position with even length.
3.  $\pi_{d(\omega)}[a, a + n] = \pi_{d(\omega)}[2b + 1, 2b + 2m]$ , odd starting position with even length.
4.  $\pi_{d(\omega)}[a, a + n] = \pi_{d(\omega)}[2b + 1, 2b + 2m - 1]$ , odd starting position with odd length.

Consider two subpermutations  $\pi_{d(\omega)}[2c, 2c + n]$  and  $\pi_{d(\omega)}[2d + 1, 2d + n + 1]$ , with  $n > 2N_k$ . The subpermutation  $\pi_{d(\omega)}[2c, 2c + n]$  will have form  $d(\omega)[2c, 2c + n - 1]$ , and  $\pi_{d(\omega)}[2d + 1, 2d + n + 1]$  will have form  $d(\omega)[2d + 1, 2d + n]$ . Since the length of these factors is at least  $2N_k$ , we know  $d(\omega)[2c, 2c + n - 1] \neq d(\omega)[2d + 1, 2d + n]$ , and thus  $\pi_{d(\omega)}[2c, 2c + n] \neq \pi_{d(\omega)}[2d + 1, 2d + n + 1]$  because they do not have the same form. Thus we can break up the set  $\text{Perm}^{d(\omega)}(n)$  into two classes of subpermutations, namely the subpermutations that start at an even position or an odd position. So say that  $\text{Perm}_{\text{ev}}^{d(\omega)}(n)$  is the set of subpermutations  $p$  of length  $n$  so that  $p = \pi_{d(\omega)}[2b, 2b + n - 1]$  for some  $b$ , and that  $\text{Perm}_{\text{odd}}^{d(\omega)}(n)$  is the set of subpermutations  $p$  of length  $n$  so that  $p = \pi_{d(\omega)}[2b + 1, 2b + n]$  for some  $b$ . Thus

$$\text{Perm}^{d(\omega)}(n) = \text{Perm}_{\text{ev}}^{d(\omega)}(n) \cup \text{Perm}_{\text{odd}}^{d(\omega)}(n),$$

where

$$\text{Perm}_{\text{ev}}^{d(\omega)}(n) \cap \text{Perm}_{\text{odd}}^{d(\omega)}(n) = \emptyset.$$

Thus for  $n \geq N_k$  and the subpermutation  $\pi_{d(\omega)}[2a, 2a + 2n - 1]$ , we see for  $p = \pi_\omega[a, a + n + k - 1]$ ,  $\delta(p) = p' = \pi_{d(\omega)}[2a, 2a + 2n - 1]$ . Thus the map

$$\delta : \text{Perm}^\omega(n + k) \mapsto \text{Perm}_{\text{ev}}^{d(\omega)}(2n)$$

is a surjective map.

For  $p = \pi_{d(\omega)}[a, a + n + k - 1]$ , we can then define three additional maps by looking at the left, right, and middle restrictions of  $\delta(p) = p'$ . These maps are

$$\delta_L : \text{Perm}^\omega(n + k) \mapsto \text{Perm}_{\text{ev}}^{d(\omega)}(2n - 1)$$

$$\delta_R : \text{Perm}^\omega(n + k) \mapsto \text{Perm}_{\text{odd}}^{d(\omega)}(2n - 1)$$

$$\delta_M : \text{Perm}^\omega(n + k) \mapsto \text{Perm}_{\text{odd}}^{d(\omega)}(2n - 2)$$

and are defined by

$$\delta_L(p) = L(\delta(p)) = L(p')$$

$$\delta_R(p) = R(\delta(p)) = R(p')$$

$$\delta_M(p) = M(\delta(p)) = M(p')$$

It can be readily verified that these three maps are surjective. To see an example of this, consider the map  $\delta_L$ , and let  $\pi_{d(\omega)}[2b, 2b + 2n - 2]$  be a subpermutation of  $\pi_{d(\omega)}$  in  $\text{Perm}_{\text{ev}}^{d(\omega)}(2n - 1)$ . Then for the subpermutation  $p = \pi_{\omega}[b, b + n + k - 1]$ ,  $\delta_L(p) = L(p') = \pi_{d(\omega)}[2b, 2b + 2n - 2]$  so  $\delta_L$  is surjective. A similar argument will show that  $\delta_R$  and  $\delta_M$  are also surjective.

**Lemma 3.5** *For  $n \geq N_k$ :*

$$\begin{aligned}\tau_{d(\omega)}(2n - 1) &\leq 2(\tau_{\omega}(n + k)) \\ \tau_{d(\omega)}(2n) &\leq \tau_{\omega}(n + k) + \tau_{\omega}(n + k + 1)\end{aligned}$$

The maps  $\delta$ ,  $\delta_L$ ,  $\delta_R$ , and  $\delta_M$  can be, but are not necessarily, injective maps. For this example we will use the Thue-Morse word  $T$ , defined in Section 6, and subpermutations of  $\pi_T$ . We will use subpermutations of length 9, with  $n = 7$  and  $k = 2$ , to keep the example subpermutations short, but examples like this (as in Corollary 3.3) can be found for subpermutations of  $\pi_T$  of length  $2^r + 1$  for any  $r \geq 3$ .

Let  $p = \pi_T[0, 8] = (4\ 9\ 7\ 2\ 6\ 1\ 3\ 8\ 5)$  and  $q = \pi_T[12, 20] = (5\ 9\ 7\ 2\ 6\ 1\ 3\ 8\ 4)$ . So  $p \neq q$  and both of these subpermutations have form  $T[0, 7] = T[12, 19] = 01101001$ . Then applying the map  $\delta$  we see:

$$p' = \delta(p) = (5\ 8\ 14\ 13\ 12\ 10\ 3\ 6\ 11\ 9\ 1\ 2\ 4\ 7) = \delta(q) = q'$$

So  $p' = q'$  which implies  $\delta_L(p) = \delta_L(q)$ ,  $\delta_R(p) = \delta_R(q)$ , and  $\delta_M(p) = \delta_M(q)$ . Thus these 4 maps are not injective in general and the values in Lemma 3.5 are only an upper bound.

## 4 Injective Restriction Mappings

In this section we will investigate when the restriction mappings are injective. If  $\delta$  is not injective, then  $\delta_R$ ,  $\delta_L$ , and  $\delta_M$  will not be injective. But when  $\delta$  is injective it implies  $\delta_R$  and  $\delta_L$  are injective in general, as shown by Proposition 4.4. Unfortunately, this does not imply that the map  $\delta_M$  is injective, as can be seen in Lemma 6.7.

**Lemma 4.1** *For the word  $\omega$ , let  $p = \pi_{\omega}[a, a + n + k - 1]$ ,  $q = \pi_{\omega}[b, b + n + k - 1]$ ,  $p'$ , and  $q'$  be as above. Suppose  $L^k(p) = L^k(q)$ , but  $\omega[a + n - 1]$  and  $\omega[b + n - 1]$  each have a different  $C_j$  class as a prefix and  $\omega[a + n - 1] < \omega[b + n - 1]$ . Then there is a  $j$  so that  $\omega[a + n - 1]$  has  $C_j$  as a prefix and  $\omega[b + n - 1]$  has  $C_{j+1}$  as a prefix. Moreover,  $|p'_{2n-2} - q'_{2n-2}| \geq 1$  and  $|p'_{2n-1} - q'_{2n-1}| \geq 1$ .*

The following definitions describe patterns which can occur within a set of subpermutations.

**Definition** A subpermutation  $p = \pi[a, a + n]$  is of type  $k$ , for  $k \geq 1$ , if  $p$  can be decomposed as

$$p = (\alpha_1 \cdots \alpha_k \lambda_1 \cdots \lambda_l \beta_1 \cdots \beta_k)$$

where  $\alpha_i = \beta_i + \varepsilon$  for each  $i = 1, 2, \dots, k$  and an  $\varepsilon \in \{-1, 1\}$ .

Some examples of subpermutations of type 1, 2, and 3 (respectively) are:

$$(2\ 3\ 5\ 4\ 1) \quad (2\ 5\ 4\ 1\ 3\ 6) \quad (3\ 7\ 5\ 1\ 2\ 6\ 4)$$

**Definition** Suppose that the subpermutation  $p = \pi[a, a+n]$  is of type  $k$  so that for  $\varepsilon \in \{-1, 1\}$ ,  $\alpha_i = \beta_i + \varepsilon$  for each  $i = 1, 2, \dots, k$ . If there exists a subpermutation  $q = \pi[b, b+n]$  of type  $k$  so that  $p$  and  $q$  can be decomposed as:

$$\begin{aligned} p &= \pi_T[a, a+n] = (\alpha_1 \cdots \alpha_k \lambda_1 \cdots \lambda_l \beta_1 \cdots \beta_k) \\ q &= \pi_T[b, b+n] = (\beta_1 \cdots \beta_k \lambda_1 \cdots \lambda_l \alpha_1 \cdots \alpha_k) \end{aligned}$$

then  $p$  and  $q$  are said to be a *complementary pair of type  $k$* . If  $p$  and  $q$  are a complementary pair of type  $k \leq 0$  then  $p = q$ .

For example, the subpermutations  $(2\ 3\ 5\ 4\ 1)$  and  $(1\ 3\ 5\ 4\ 2)$  are a complementary pair of type 1.

**Lemma 4.2** *For the word  $\omega$ , let  $p, q, p'$ , and  $q'$  be as above, then  $p'$  and  $q'$  are not a complementary pair of type 1.*

**Claim 4.3** *Suppose  $f$  is a restriction map, so either  $f = R$ ,  $f = L$ , or  $f = M$ . If  $f(p') = f(q')$  then  $d(u) = d(v)$ .*

We are now to the main result of this section. We show that when  $\delta$  is injective we find that both of  $\delta_L$  and  $\delta_R$  are injective.

**Proposition 4.4** *For the word  $\omega$ , let  $p, q, p'$ , and  $q'$  be as above. Then*

- (a)  $p' = q'$  if and only if  $R(p') = R(q')$ .
- (b)  $p' = q'$  if and only if  $L(p') = L(q')$ .

Therefore when  $\delta$  is injective,  $\delta_R$  and  $\delta_L$  are both injective as well. A troubling fact is the map  $\delta$  being injective does not imply  $\delta_M$  is injective. As will be shown for the Thue-Morse word  $T$ , there are cases of distinct subpermutations  $p$  and  $q$  where  $\delta(p) \neq \delta(q)$  but  $\delta(p)_M = \delta_M(q)$ . The following sections deal with some different words and we will show when  $\delta$  and  $\delta_M$  are injective, but these proofs will use special properties of the words considered.

## 5 Sturmian Words

In this section we will investigate the permutation complexity of Sturmian words under the doubling map. An infinite word  $s$  is a *Sturmian word* if for each  $n \geq 0$ ,  $s$  has exactly  $n+1$  distinct factors of length  $n$ , or  $\rho_s(n) = n+1$  (the only factor of length  $n=0$  being the empty-word). The class of Sturmian words have been a topic of much study (see [3, 5, 7]). An equivalent definition for Sturmian words is that they are the class of aperiodic balanced binary words. A word is *balanced* if for all factors  $u$  and  $v$  with  $|u| = |v|$ ,  $||u|_a - |v|_a| \leq 1$  for each  $a$  in the alphabet.

First we will show when the map  $\delta$  is applied to permutations from a Sturmian word,  $\delta$  is injective and thus a bijection. Then we show the maps  $\delta_R$ ,  $\delta_L$ , and  $\delta_M$  are injective as well and thus also bijections. First we look at the permutation complexity of the Sturmian words which has been calculated.

**Lemma 5.1** ([10]) *Let  $s$  be a Sturmian word. For natural numbers  $a_1$  and  $a_2$  we have  $\pi_s[a_1, a_1+n+1] = \pi_s[a_2, a_2+n+1]$  if and only if  $s[a_1, a_1+n] = s[a_2, a_2+n]$ .*

**Theorem 5.2** ([10]) *Let  $s$  be a Sturmian word. For each  $n \geq 2$ ,  $\tau_s(n) = n$ .*



Fix a Sturmian word  $s$  over  $\{0, 1\}$ . Since  $s$  is balanced, there is some  $k > 0$  so that for  $\alpha, \beta \in \{0, 1\}$ , with  $\alpha \neq \beta$ , every  $\alpha$  is followed by either  $k$  or  $k - 1$   $\beta$ 's. So consecutive  $\alpha$ 's will look like either  $\alpha\beta^k\alpha$  or  $\alpha\beta^{k-1}\alpha$ . Let  $d(s)$  be the image of  $s$  under the doubling map. Then  $\pi_s$  is the infinite permutation associated to  $s$ , and  $\pi_{d(s)}$  is the infinite permutation associated to  $d(s)$ .

We will now calculate the permutation complexity of  $d(s)$ . By Lemma 2.3 we may assume there is a natural number  $k > 0$  so that each 1 is followed by either  $0^k1$  or  $0^{k-1}1$ , because  $d(s)$  and  $d(\bar{s})$  have the same permutation complexity. There will be  $k + 1$  classes of factors of  $s$ , which are  $C_0 = 0^k$ ,  $C_1 = 0^{k-1}1$ ,  $\dots$ ,  $C_{k-1} = 01$ ,  $C_k = 10$ . Since Sturmian words are uniformly recurrent ([5]), there is an  $N \in \mathbb{N}$  so that each factor of  $s$  of length  $n \geq N_s$  will contain each of  $C_0, C_1, \dots, C_k$ . The map  $\delta$  is injective, and thus bijective, when applied to subpermutations associated with a Sturmian words.

**Lemma 5.3** *For the Sturmian word  $s$ , and subpermutations  $p = \pi_s[a, a + n + k - 1]$  and  $q = \pi_s[b, b + n + k - 1]$  of length  $n \geq N_s$ ,  $p = q$  if and only if  $\delta(p) = \delta(q)$ .*

When Lemma 5.3 is used with Proposition 4.4 we see the maps  $\delta_L$  and  $\delta_R$  are also injective, and thus are bijections. The map  $\delta_M$  is also injective when applied to subpermutations associated with a Sturmian words.

**Lemma 5.4** *For the Sturmian word  $s$ , and subpermutations  $p = \pi_s[a, a + n + k - 1]$  and  $q = \pi_s[b, b + n + k - 1]$  of length  $n \geq N_s$ ,  $p = q$  if and only if  $M(p') = M(q')$ .*

The following theorem will give the permutation complexity of the image of a Sturmian word under the letter doubling map.

**Theorem 5.5** *Let  $s$  be a Sturmian word over  $\mathcal{A}$ , where for  $\alpha, \beta \in \mathcal{A}$ ,  $\alpha \neq \beta$ , there are strings of either  $k$  or  $k - 1$   $\alpha$  between each  $\beta$ . There is an  $N$  so that each factor of  $s$  of length at least  $N$  will contain each of  $\alpha^k, \alpha^{k-1}\beta, \dots, \alpha\beta, \beta$ . For each  $n \geq 2N$  the permutation complexity of  $d(s)$  is*

$$\tau_{d(s)}(n) = n + 2k + 1$$

## 6 Thue-Morse Word

In this section we will investigate the permutation complexity of  $d(T)$ , the image of the Thue-Morse word,  $T$ , under the doubling map,  $d$ . The Thue-Morse word is:

$$T = 01101001100101101001011001101001 \dots,$$

and the Thue-Morse morphism is:

$$\mu_T : 0 \rightarrow 01, \quad 1 \rightarrow 10.$$

This word was introduced by Axel Thue in his studies of repetitions in words ([12]). For a more in depth look at further properties, independent discoveries, and applications of the Thue-Morse word see [1].

The factor complexity of the Thue-Morse word was computed independently by two groups in 1989 ([4] and [8]). The calculation of the permutation complexity of  $d(T)$  will use the formula for the factor complexity of  $T$ . We will use the formula calculated by S. Brlek.

**Proposition 6.1** ([4]) *For  $n \geq 3$ , the function  $\rho_T(n)$  is given by*

$$\rho_T(n) = \begin{cases} 6 \cdot 2^{r-1} + 4p & 0 < p \leq 2^{r-1} \\ 8 \cdot 2^{r-1} + 2p & 2^{r-1} < p \leq 2^r \end{cases}$$

where  $r$  and  $p$  are uniquely determined by the equation  $n = 2^r + p + 1$ , with  $0 < p \leq 2^r$ .

Let  $\pi_T$  be the infinite permutation associated to the Thue-Morse word  $T$ . In [13], the permutation complexity of  $T$  was calculated.

**Theorem 6.2** ([13]) *For any  $n \geq 6$ , where  $n = 2^r + p$  with  $0 < p \leq 2^r$ ,*

$$\tau_T(n) = 2(2^{r+1} + p - 2).$$

We will now investigate the permutation complexity of  $d(T)$ . To begin, we consider complementary pairs which occur in  $\pi_T$ .

**Theorem 6.3** ([13]) *Let  $p$  and  $q$  be distinct subpermutations of  $\pi_T$ . Then  $p$  and  $q$  have the same form if and only if  $p$  and  $q$  are a complementary pair of type  $k$ , for some  $k \geq 1$ .*

The left and right restrictions preserve complementary pairs of type  $k \geq 2$ , and middle restrictions preserve complementary pairs of type  $k \geq 3$ . Proposition 6.4 follows directly from [13], Proposition 4.1. We then see when complementary pairs of type  $k$  can occur, for each  $k \geq 0$ .

**Proposition 6.4** ([13]) *Suppose  $p = \pi_T[a, a + n]$  and  $q = \pi_T[b, b + n]$  are a complementary pair of type  $k \geq 1$ .*

- (a)  $L(p)$  and  $L(q)$  are a complementary pair of type  $k - 1$ .
- (b)  $R(p)$  and  $R(q)$  are a complementary pair of type  $k - 1$ .
- (c)  $M(p)$  and  $M(q)$  are a complementary pair of type  $k - 2$ .

**Proposition 6.5** ([13]) *Let  $n > 4$  be a natural number and let  $p$  and  $q$  be subpermutations of  $\pi_T$  of length  $n + 1$  with the same form. There exist  $r$  and  $c$  so that  $n = 2^r + c$ , where  $0 \leq c < 2^r$ .*

- (a) *If  $0 \leq c < 2^{r-1} + 1$ , then either  $p = q$  or  $p$  and  $q$  are a complementary pair of type  $c + 1$ .*
- (b) *If  $2^{r-1} + 1 \leq c < 2^r$ , then  $p = q$ .*

Now to calculate the permutation complexity of  $d(T)$  we need to identify the classes of factors of  $T$  with blocks of the same letter. Since  $T$  is overlap-free, and thus cube free, we can identify the 4 classes of factors of  $T$ , which are  $C_0 = 00$ ,  $C_1 = 01$ ,  $C_2 = 10$ , and  $C_3 = 11$ . For each  $i \in \mathbb{N}$ ,  $T[i] = T_i T_{i+1} \dots$  will have exactly one the above classes of words as a prefix. Since the Thue-Morse word is uniformly recurrent ([1]), there is an  $N \in \mathbb{N}$  so that each factor of  $T$  of length  $n \geq N$  will contain each of  $C_0$ ,  $C_1$ ,  $C_2$ , and  $C_3$ . It is readily verified that any factor of length  $n \geq 9$  will contain these 4 classes of words.

Let  $u = T[a, a + n - 1]$  and  $v = T[b, b + n - 1]$ ,  $a \neq b$ , be factors of  $T$  of length  $n \geq 9$ , so  $C_j$  is a factor of both  $u$  and  $v$  for each  $0 \leq j \leq 3$ . Let  $p = \pi_T[a, a + n + 1]$  and  $q = \pi_T[b, b + n + 1]$  be subpermutations of  $\pi_T$ . Then define subpermutations  $\delta(p) = p' = \pi_{d(T)}[2a, 2a + 2n - 1]$  and  $\delta(q) = q' = \pi_{d(T)}[2b, 2b + 2n - 1]$  as in Proposition 3.2, with  $k = 2$ . The following lemma concerns the relationship of  $p$  and  $q$  to  $p'$  and  $q'$ .

**Lemma 6.6** *Let  $p$  and  $q$  be subpermutations of length  $n + 2$  of  $\pi_T$ , with  $n \geq 9$ , and let  $p' = \delta(p)$  and  $q' = \delta(q)$ .*

- (a) *If  $n \neq 2^r - 1$  or  $2^r$  for any  $r \geq 3$ ,  $p = q$  if and only if  $p' = q'$ .*
- (b) *If  $n = 2^r - 1$  or  $2^r$  for some  $r \geq 3$ ,  $p$  and  $q$  have the same form if and only if  $p' = q'$ .*

Thus, for  $n \geq 9$ , the maps  $\delta$ ,  $\delta_L$ , and  $\delta_R$  when applied to permutations associated with the Thue-Morse word are injective when  $n \neq 2^r - 1$  or  $2^r$  for any  $r \geq 3$ , so  $|\text{Perm}_{ev}^{d(T)}(2n)| = |\text{Perm}^T(n + 2)|$ ,  $|\text{Perm}_{ev}^{d(T)}(2n - 1)| = |\text{Perm}^T(n + 2)|$ , and  $|\text{Perm}_{odd}^{d(T)}(2n - 1)| = |\text{Perm}^T(n + 2)|$ .

When  $n = 2^r - 1$  or  $2^r$  for some  $r \geq 3$  the maps  $\delta$ ,  $\delta_R$ , and  $\delta_L$  are surjective, but not injective because complementary pairs of type 1 or 2 will give the same subpermutation under  $\delta$ . In this case, if  $p$  and  $q$  are subpermutations of  $\pi_T$  of length  $n + 2$ , where  $p$  has form  $u$  and  $q$  has form  $v$ ,  $|u| = |v| = n + 1$ ,  $\delta(p) = \delta(q)$  if and only if  $u = v$ . Thus with Proposition 4.4 we see  $\delta_L(p) = \delta_L(q)$  and  $\delta_R(p) = \delta_R(q)$  if and only if  $u = v$ . Thus the number of subpermutations of  $\pi_{d(T)}$  for these lengths are determined by the number of factors of  $T$ , so  $|\text{Perm}_{ev}^{d(T)}(2n)| = |\mathcal{F}_T(n + 1)|$ ,  $|\text{Perm}_{ev}^{d(T)}(2n - 1)| = |\mathcal{F}_T(n + 1)|$ , and  $|\text{Perm}_{odd}^{d(T)}(2n - 1)| = |\mathcal{F}_T(n + 1)|$ .

The following lemma shows when the map  $\delta_M$  is injective when applied to permutations associated with the Thue-Morse word.

**Lemma 6.7** *For the Thue-Morse word  $T$ , let  $p, q, p'$ , and  $q'$  be as above. Then*

- (a) *If  $n \neq 2^r - 1, 2^r$ , or  $2^r + 1$  for any  $r \geq 3$ ,  $p' = q'$  if and only if  $M(p') = M(q')$ .*
- (b) *If  $n = 2^r - 1, 2^r$ , or  $2^r + 1$  for some  $r \geq 3$ ,  $p$  and  $q$  have the same form if and only if  $M(p') = M(q')$ .*

Thus, for  $n \geq 9$ , the map  $\delta_M$  when applied to permutations associated with the Thue-Morse word are injective when  $n \neq 2^r - 1, 2^r$ , or  $2^r + 1$  for any  $r \geq 3$ , so  $|\text{Perm}_{odd}^{d(T)}(2n - 2)| = |\text{Perm}^T(n + 2)|$ .

When  $n = 2^r - 1, 2^r$ , or  $2^r + 1$  for some  $r \geq 3$  the map  $\delta_M$  is surjective, but not injective. In this case, if  $p$  and  $q$  are subpermutations of  $\pi_T$  of length  $n + 2$ , where  $p$  has form  $u$  and  $q$  has form  $v$ ,  $|u| = |v| = n + 1$ ,  $\delta_M(p) = \delta_M(q)$  if and only if  $u = v$ . Thus the number of subpermutations of  $\pi_{d(T)}$  of length  $2n - 2$  which start in an odd position are determined by the number of factors of  $T$  of length  $n + 1$ , so  $|\text{Perm}_{odd}^{d(T)}(2n - 2)| = |\mathcal{F}_T(n + 1)|$ .

We are now ready to calculate the permutation complexity of  $d(T)$ .

**Theorem 6.8** *For the Thue-Morse word  $T$ , let  $n \geq 9$ .*

- (a) *If  $n = 2^r$ , then*

$$\tau_{d(T)}(2n - 1) = 2^{r+2} + 2^{r+1}$$

$$\tau_{d(T)}(2n) = 2^{r+2} + 2^{r+1} + 4$$

- (b) *If  $n = 2^r + p$  for some  $0 < p \leq 2^r - 1$ , then*

$$\tau_{d(T)}(2n - 1) = 2^{r+3} + 4p$$

$$\tau_{d(T)}(2n) = 2^{r+3} + 4p + 2$$

## References

- [1] J.-P. Allouche & J. Shallit (1999): *The Ubiquitous Prouhet-Thue-Morse Sequence*. In: *Sequences and Their Applications, Proc. SETA'98*, Springer-Verlag, pp. 1–16.
- [2] S.V. Avgustinovich, A.E. Frid, T. Kamae & P.V. Salimov (2009): *Infinite permutations of lowest maximal pattern complexity*. Available at <http://arxiv.org/abs/0910.5696v2>.
- [3] J. Berstel (1995): *Recent Results In Sturmian Words*. In J. Dassaw, editor: *Developments in Language Theory*, World Scientific, Singapore.
- [4] S. Brlek (1989): *Enumeration of factors in the Thue-Morse word*. *Discrete Appl. Math* 24, pp. 83–96, doi:10.1016/0166-218X(92)90274-E.

- [5] E.M. Coven & G.A. Hedlund (1973): *Sequences with minimal block growth*. *Math. Systems Theory* 7(2), pp. 138–153, doi:10.1007/BF01762232.
- [6] D.G. Fon-Der-Flaass & A.E. Frid (2007): *On periodicity and low complexity of infinite permutations*. *European J. Combin.* 28(8), pp. 2106–2114, doi:10.1016/j.ejc.2007.04.017.
- [7] M. Lothaire (2002): *Algebraic Combinatorics on Words*. *Encyclopedia of Mathematics and its Applications* 90, Cambridge University Press.
- [8] A.de Luca & S. Varricchio (1989): *Some combinatorial properties of the Thue-Morse sequence and a problem in semigroups*. *Theoret. Comput. Sci.* 63, pp. 333–348, doi:10.1016/0304-3975(89)90013-3.
- [9] M.A. Makarov (2006): *On permutations generated by infinite binary words*. *Sib. Elektron. Mat. Izv.* 3, pp. 304–311. (in Russian).
- [10] M.A. Makarov (2009): *On the permutations generated by the Sturmian Words*. *Sib. Math. J.* 50(3), pp. 674–680, doi:10.1007/s11202-009-0076-6.
- [11] M.A. Makarov (2010): *On the infinite permutation generated by the period doubling word*. *European J. Combin.* 31(1), pp. 368–378, doi:10.1016/j.ejc.2009.03.038.
- [12] A. Thue (1912): *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen*. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* 1, pp. 1–67.
- [13] S. Widmer (2011): *Permutation Complexity of the Thue-Morse Word*. *Adv. in Appl. Math.* 47(2), pp. 309 – 329, doi:10.1016/j.aam.2010.08.002.